Almost Invariance and Noninteracting Control: A Frequency-Domain Analysis

H. L. Trentelman and J. W. van der Woude
Department of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513
5600 MB Eindhoven, The Netherlands

Submitted by Paul A. Fuhrmann

ABSTRACT

We solve a number of feedback synthesis problems in the context of noninteracting control or block-diagonal decoupling for finite-dimensional linear time-invariant systems. We consider a plant that, apart from a control input and a measurement output, has a given number of exogenous input vectors and the same number of exogenous output vectors. The decoupling problem studied is to find dynamic compensators from the plant measurement output (which in this paper will be assumed to be the full plant state) to the plant control input in such a way that the following requirements are met: (1) the closed-loop transfer matrix is block-diagonal, (2) the remaining diagonal blocks are stable with respect to an *a priori* given first stability set, and (3) the closed-loop system is internally stable with respect to an *a priori* given second (in general larger) stability set. In addition, we study the "almost" version of the above problem. In the latter the requirement of exact decoupling is replaced by a requirement of approximate decoupling in the sense that the compensators to be designed should yield off-diagonal blocks in the closed-loop transfer matrix that are arbitrarily small in H^{∞}-norm. Necessary and sufficient conditions for the existence of such dynamic compensators are formulated in terms of controlled invariant and almost controlled invariant subspaces.

1. INTRODUCTION

This paper deals with a number of feedback synthesis problems that appear in the context of noninteracting control or (block) diagonal decoupling for finite-dimensional linear time-invariant systems. Over the past

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twenty-five or so years a considerable number of papers on this subject have appeared in the control-theory literature. For excellent overviews of the existing literature we refer to [7] or [5]. The setup in the present paper will differ fundamentally from the one that is usually considered in the literature. We want to make clear from the outset that the purpose of this paper is *not* to present a new contribution to the "classical" problem of noninteracting control as studied in the above references, but to formulate and resolve a number of *new* synthesis problems in the noninteracting-control context. These new synthesis problems are in principle independent of the existing problem formulations. The alternative point of view towards noninteracting control as adopted in the present paper was initiated in [16], where also some preliminary results concerning the synthesis problems to be considered here can be found.

Following [16], we shall consider a plant that, apart from a control input and a measurement output (which in this paper will always be assumed to be the full state of the plant), has a given number of exogenous inputs and the same number of exogenous outputs. Basically, the problem of noninteracting control that will be considered here is to design a dynamic feedback compensator from the measured plant output to the plant control input in such a way that the resulting closed-loop system is block-diagonal, with the sizes of the blocks compatible with the *a priori* given dimensions of the exogenous inputs and exogenous outputs. Stated differently: it is required to design an automatic feedback mechanism in such a way that in the closed-loop system the existing interaction between the exogenous variables is eliminated and to make sure that these variables influence each other only one at a time. An illustration of this setup is given in Figure 1.

The most important feature that distinguishes the abovementioned setup from the classical one is that in this formulation the exogenous inputs are specified beforehand, while in the classical case it is part of the problem to design these inputs. More precisely, the classical problem of noninteracting control can be roughly stated as follows: given a plant with a control input, a

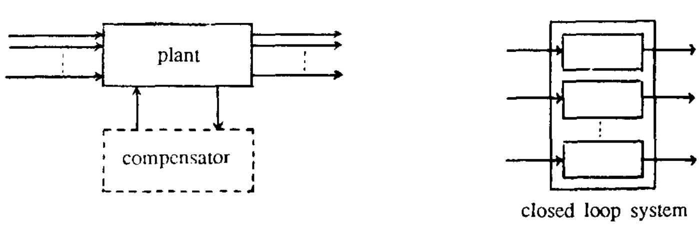

FIG. 1.

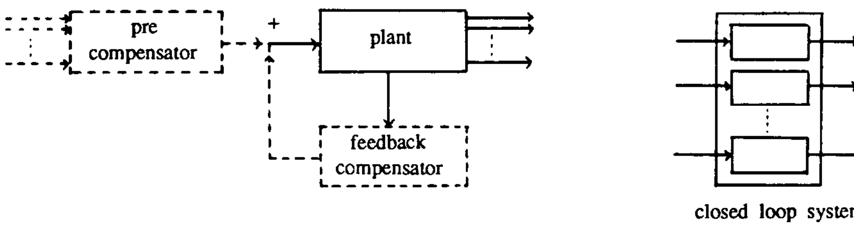

FIG. 2.

measurement output, and a given number of exogenous outputs, design exogenous input variables, a precompensator having these exogenous inputs as input variables, and finally a compensator from the measured output to the plant control input such that the closed-loop system as specified in Figure 2 is block-diagonal. Additionally, in order to avoid trivialities some typical requirements on output controllability or functional reproducability of the closed-loop system are imposed. Requiring both the precompensator and the feedback compensator to be static then yields the so-called restricted decoupling problem (RDP) [17], while allowing both compensators to be dynamic yields the extended decoupling problem (EDP) [17] (as explained in [5]).

In our opinion both of the main problem formulations as stated above are useful in the context of noninteracting-control design. For some reason, however, the former one is badly neglected in the control-theory literature, though its formulation appears to be a very natural one. In this paper we shall try to fill up this gap. We shall present an extensive treatment of the problem, including several stability issues. Moreover, the natural extension of the problem to the context of *almost* block-diagonal decoupling will also be treated. In the latter problem the off-diagonal blocks are not required to be exactly equal to zero but can be made arbitrarily small in H^∞-norm.

Roughly speaking, this paper is divided into two main parts, of which the first deals with the *exact* version of the noninteracting control problem as sketched above, and the second with its *almost* version. The main contribution of the first part is a result that gives necessary and sufficient conditions for solvability of the (exact) noninteracting-control problem in a rather general formulation. Apart from block-diagonal decoupling, this formulation requires *internal stability* of the closed-loop system with respect to a first stability set, and at the same time *input-output stability* of the diagonal blocks with respect to a second, possibly smaller, stability set. The main contribution of the second part of the paper is a result that gives necessary and sufficient conditions for solvability of the "almost" analogue of the abovementioned problem. If we take all stability sets involved to be equal to the entire complex plane, in both the exact and the almost version we reobtain the conditions found in [16] as special cases. (We do however note that in [16] no proof was given of the solvability conditions for the almost noninteracting-control problem.)

The approach that will be adopted in this paper consists of a mixture of frequency-domain concepts and concepts originating from the geometric approach to linear systems. An important role will be played by some typical *controlled invariant* and *almost controlled invariant subspaces*. These subspaces will be studied mainly in terms of their frequency-domain characterizations, in particular in terms of (ξ, ω) representations. This concept was introduced in [4] and elaborated further in [8] and [11]. Typically in this paper, *solvability conditions* for the various synthesis problems will be given in terms of controlled invariant and almost controlled invariant subspaces, while the *constructions* of the actual dynamic compensators are based directly on the (ξ, ω)-representation descriptions of these subspaces.

The paper is organized as follows. Sections 2 to 5 deal with the exact version of the noninteracting-control problem. In Section 2 the main problem formulations are collected. In Sections 3 and 4 some preliminary results with respect to these problems are derived, and in Section 5 the main results can be found. Sections 6 to 8 deal with the almost versions of the problems defined in Section 2. These are stated in Section 6. Section 7 gives some preliminary results on these problems, and finally in Section 8 the main results can be found. Some of the proofs are deferred to Appendices A, B and C.

2. NONINTERACTING CONTROL: PROBLEM FORMULATION

Consider the finite-dimensional linear time-invariant system

$$\Sigma \qquad \dot{x}(t) = Ax(t) + Bu(t) + \sum_{i=1}^{k} G_i v_i(t), \tag{2.1a}$$

$$z_i(t) = D_i x(t), \qquad i \in \underline{k}, \tag{2.1b}$$

with $x(t) \in \mathbb{R}^n =: X$ the state of the system, $u(t) \in \mathbb{R}^m =: U$ the control input, $v_i(t) \in \mathbb{R}^{q_i} =: V_i$ the ith exogenous input, and $z_i(t) \in \mathbb{R}^{p_i} =: Z_i$ the ith exogenous output. k is assumed to be an integer larger than 1, and the symbol $\underline{k}$ denotes the set $\{1, 2, \ldots, k\}$. In the above $A: X \to X$, $B: U \to X$ as well as $G_i: V_i \to X$ and $D_i: X \to Z_i$ are linear maps. As a standing assumption B will be injective.

We shall be concerned with the design of dynamic compensators described by

$$\Sigma_c \qquad \dot{w}(t) = Kw(t) + Lx(t), \tag{2.2a}$$

$$u(t) = Mw(t) + Nx(t), \tag{2.2b}$$

with $w(t) \in \mathbb{R}^l =: W$ the state of the compensator and $K: W \to W$, $L: X \to W$, $M: W \to U$, and $N: X \to U$ linear maps. The dimension l of the state space W will be denoted by $\dim \Sigma_c$. The feedback interconnection of Σ with Σ_c is a system with $(v_1, v_2, \dots, v_k)$ as its input and $(z_1, z_2, \dots, z_k)$ as its output and is described by the equations

$$\dot{x}_e(t) = A_e x_e(t) + \sum_{i=1}^{k} G_{i,e} v_i(t), \tag{2.3a}$$

$$z_i(t) = D_{i,e} x_e(t), \qquad i \in \underline{k}, \tag{2.3b}$$

where we have denoted

$$x_e = \begin{pmatrix} x \\ w \end{pmatrix}, \qquad A_e = \begin{pmatrix} A + BN & BM \\ L & K \end{pmatrix}, \qquad G_{i,e} = \begin{pmatrix} G_i \\ 0 \end{pmatrix}, \qquad D_{i,e} = \begin{pmatrix} D_i & 0 \end{pmatrix}.$$

We shall denote by T the transfer matrix of the closed-loop system (2.3). T is equal to the composite matrix (T_{ij}), where

$$T_{ij}(s) = D_{j,e}(Is - A_e)^{-1} G_{i,e}, \qquad i, j \in \underline{k}, \tag{2.4}$$

represents the transfer matrix between the ith input v_i and the jth output z_j. In [16] the following problem was introduced:

PROBLEM I (Noninteracting control). Problem I is said to be solvable if there exists a compensator Σ_c such that $T_{ij} = 0$ for all $i, j \in \underline{k}$ with $i \neq j$.

If a compensator Σ_c is such that $T_{ij} = 0$ for all $i \neq j$, then it will be said to *achieve noninteraction*. In that case the resulting closed-loop transfer matrix is block diagonal:

$$T = \operatorname{blockdiag}(T_{11}, \dots, T_{kk}).$$

An important issue here will be stability. In the sequel a subset $\mathbb{C}_g$ of $\mathbb{C}$ will be called symmetric if $\mathbb{C}_g \cap \mathbb{R} \neq \varnothing$ and if it satisfies $\lambda \in \mathbb{C}_g \Leftrightarrow \bar{\lambda} \in \mathbb{C}_g$. A rational matrix will be called $\mathbb{C}_g$-stable (or g-stable) if its poles lie in $\mathbb{C}_g$. If, apart from noninteraction, we require input-output stability of the closed-loop transfer matrix from $(v_1, v_2, \ldots, v_k)$ to $(z_1, z_2, \ldots, z_k)$, we arrive at the following problem:

PROBLEM II (Noninteracting control with input-output stability). Given a symmetric subset $\mathbb{C}_g$ of $\mathbb{C}$, Problem II is said to be solvable if there exists a compensator Σ_c that achieves noninteraction such that T_{ii} is g-stable for all $i \in \underline{k}$.

A different stability issue is that of *internal stability* of the closed-loop system. Of course, if we succeed in finding a dynamic compensator that achieves noninteraction with input-output stability, due to the presence of uncontrollable and unobservable modes this does in general not mean that the closed-loop system is internally stable [in the sense that $\sigma(A_e) \subset \mathbb{C}_g$].

PROBLEM III (Noninteracting control with internal stability). Given a symmetric subset $\mathbb{C}_g$ of $\mathbb{C}$, Problem III is said to be solvable if there exists a compensator Σ_c that achieves noninteraction such that $\sigma(A_e) \subset \mathbb{C}_g$.

In this paper, as it should, input-output stability and internal stability will be treated as two different requirements. Correspondingly we will specify *two* stability sets $\mathbb{C}_f$ and $\mathbb{C}_s$. Combining the two notions, we shall require that the decoupled system be input-output stable with respect to the stability set $\mathbb{C}_f$ and internally stable with respect to the stability set $\mathbb{C}_s$. Typically, this corresponds to requiring a fast response of the to-be-controlled output variables $z_1, z_2, \ldots, z_k$ and allowing a slower response of the internal part of the system (see also [4] or [10]). Formalizing this, we arrive at the following version of the noninteracting-control problem:

PROBLEM IV (Noninteracting control with input-output and internal stability). Given two symmetric subsets $\mathbb{C}_f \subset \mathbb{C}_s$ of $\mathbb{C}$, Problem IV is said to be solvable if there exists a compensator Σ_c that achieves noninteraction such that T_{ii} is f-stable for all $i \in \underline{k}$ and $\sigma(A_e) \subset \mathbb{C}_s$.

It is our purpose to establish necessary and sufficient conditions for the solvability of the above four problems that can be checked constructively. Clearly, once we have established these for Problem IV we are done, since Problems II and III may be obtained as special cases of Problem IV by taking $\mathbb{C}_f = \mathbb{C}_g$, $\mathbb{C}_s = \mathbb{C}$, and $\mathbb{C}_f = \mathbb{C}_s = \mathbb{C}_g$ respectively. Obviously, Problem I requires only $\mathbf{C}_f = \mathbf{C}_s = \mathbf{C}$. Thus, instead of considering each problem separately, we shall concentrate on deriving conditions for solvability of Problem IV.

3. SOME GEOMETRIC CONCEPTS

Given a system (A, B) with state space X and a subspace K of X, we shall denote by $V^*(K)$ the supremal controlled invariant subspace contained in K. If $\mathbf{C}_g$ is a symmetric subset of $\mathbf{C}$, then $V_g^*(K)$ will denote the supremal stabilizability subspace in K [4]. If instead of one we specify two stability sets $\mathbf{C}_f$ and $\mathbf{C}_s$, then $V_f^*(K)$ and $V_s^*(K)$ will denote the supremal stabilizability subspace with respect to $\mathbf{C}_f$ and $\mathbf{C}_s$ respectively. The system (A, B) will be called g-stabilizable (s-stabilizable) if it is stabilizable with respect to $\mathbf{C}_g$ ($\mathbf{C}_s$). A similar terminology will be used in the context of detectability.

If ξ and ω are an n-vector and an m-vector of real rational functions respectively and if $x_0 \in X$, then the expression

$$x_0 = (Is - A)\xi(s) - B\omega(s) \tag{3.1}$$

will be called a (ξ, ω) representation of x_0. A (ξ, ω) representation will be called *regular* if both ξ and ω are strictly proper. Given a symmetric subset $\mathbf{C}_g$ of $\mathbf{C}$, a (ξ, ω) representation will be called g-stable if ξ is g-stable. Assuming that B is injective, this implies that also ω is g-stable. The notion of (ξ, ω) representation can be used to give frequency-domain characterizations of the various controlled invariant and almost controlled invariant subspaces appearing in the literature on the geometric approach to linear systems (see e.g. [4], [8], or [11]). In particular, if $\mathbf{C}_g$ is a symmetric subset of $\mathbf{C}$ and if K is a subspace of X such that $K = \ker H$, then we have

$$V_g^*(K) = \{x_0 \in X \mid x_0 \text{ has a regular } g\text{-stable } (\xi, \omega) \text{ representation with } H\xi = 0\}. \tag{3.2}$$

In our considerations on noninteracting control with input-output and internal stability, an important role will be played by controlled invariant subspaces that, instead of on one subspace K and one stability set $\mathbf{C}_g$, rather depend on a pair of subspaces and a pair of stability sets. In the following, let K_1 and K_2 be subspaces of X such that $K_2 \subset K_1$. Let H_1 and H_2 be linear maps such that $K_i = \ker H_i$. Let $\mathbf{C}_f \subset \mathbf{C}_s$ be symmetric subsets of $\mathbf{C}$. The following definition slightly generalizes [4, Definition 4.2]:

DEFINITION 3.1.

$$V_{f,s}(K_1,K_2) := \{x_0 \in X \mid x_0 \text{ has a regular } s\text{-stable } (\xi,\omega)\text{-representation with } H_1\xi = 0 \text{ and } H_2\xi\ f\text{-stable}\}.$$

Clearly, this defines a linear subspace of X which, by (3.2), is contained in $V_s^*(K_1)$, the supremal s-stabilizability subspace in K_1. Using the fact that $K_2 \subset K_1$, it is also clear that the above-defined subspace contains $V_f^*(K_2)$, the supremal f-stabilizability subspace in K_2. We note that if in the above we take $H_1 = 0$ and $H_2 = H$, then $V_{f,s}(K_1,K_2) = V_{f,s}(X,\ker H)$, the subspace of all $x_0 \in X$ having a regular s-stable (ξ,ω) representation with $H\xi$ f-stable. The latter coincides with the subspace S_{12} as studied in [4, p. 707]. In fact, by slightly adapting the proof of [4, Theorem 4.3], we arrive at the following representation of $V_{f,s}(K_1,K_2)$ (see also [12, Theorem 4.5]):

THEOREM 3.2.

$$V_{f,s}(K_1,K_2) = V_f^*(K_1) + V_s^*(K_2).$$

In the considerations on noninteracting control to come, the following observation will turn out to be instrumental:

LEMMA 3.3. *Let E be a linear subspace of X, and let E be a linear map such that $\operatorname{im} E = \mathsf{E}$. Then (A,B) is s-stabilizable and $\mathsf{E} \subset V_{f,s}(K_1,K_2)$ if and only if there exist strictly proper s-stable real rational matrices X and U such that $(Is - A)X(s) - BU(s) = I$, $H_1XE = 0$, and H_2XE is f-stable.*

Proof. $\Rightarrow$: Without loss of generality, assume that E is injective. Let $E: V \to X$, with $V = \mathbb{R}^q$. By applying the definition of $V_{f,s}$ to the vectors Ee_i, where e_i is the ith column of the $q \times q$ identity matrix, we find strictly proper s-stable real rational matrices X_1 and U_1 such that $E = (Is - A)X_1(s) - BU_1(s)$, $H_1X_1 = 0$, and H_2X_1 is f-stable. Let E' be a linear map such that $E \vdots E'$ is bijective. Since (A,B) is s-stabilizable, there are strictly proper s-stable real rational matrices X_2 and U_2 such that $E' = (Is - A)X_2(s) - BU_2(s)$ (see [4, Corollary 2.20]). Now define $X := (X_1 \vdots X_2)(E \vdots E')^{-1}$ and $U := (U_1 \vdots U_2)(E \vdots E')^{-1}$. Then $(Is - A)X(s) - BU(s) = I$. Moreover, $H_1XE = H_1X_1 = 0$, $H_2XE = H_2X_1$ is f-stable, and X and U are s-stable and strictly proper.

$\Leftarrow$: Let $Ev \in \mathsf{E}$. By taking $\xi := XEv$ and $\omega := UEv$ we obtain a (ξ, ω) representation of Ev with the properties required in Definition 3.1. The fact that (A, B) is s-stabilizable follows from [4, Corollary 2.20]. ■

Note that by taking $\mathbb{C}_f = \mathbb{C}_s$ and $K_i = K = \ker H$ in the above lemma, we find that (A, B) is s-stabilizable and $\operatorname{im} E \subset V_s^*(K)$ if and only if there exist strictly proper s-stable real rational matrices X and U such that $(IS - A)X(s) - BU(s) = I$ and $HXE = 0$. Finally note that we do not need Theorem 3.2 in the proof of Lemma 3.3. The importance of Theorem 3.2 is that it enables us to calculate explicitly the subspace $V_{f,s}(K_1, K_2)$, for example by using the construction in [17, p. 114].

4. INPUT-OUTPUT DESCRIPTION OF INTERNAL STABILITY

One of the requirements that should be met by the compensator Σ_c to be designed is that the closed-loop system is internally stable, i.e. that $\sigma(A_e) \subset \mathbb{C}_s$. In this section we shall see that this requirement has an equivalent formulation in terms of certain transfer matrices associated with the closed-loop system. Using this fact, we shall show that every pair of strictly proper s-stable real rational matrices (X, U) such that $(Is - A)X(s) - BU(s) = I$ gives rise to an s-stabilizing dynamic compensator $U(s)X(s)^{-1}$.

Consider the system (A, B), and let its input to state transfer matrix be given by

$$P(s) := (Is - A)^{-1}B. \tag{4.1}$$

Let the transfer matrix of the compensator (2.2) be given by

$$F(s) := N + M(Is - K)^{-1}L. \tag{4.2}$$

The following result states that the internal stability of the closed system formed by interconnecting the system $\dot{x} = Ax + Bu$ with the compensator (2.2) can be characterized in terms of expressions involving their transfer matrices:

LEMMA 4.1. *Assume that (A, B) is s-stabilizable, (K, L) is s-stabilizable, and (M, K) is s-detectable. Then we have*

$$\sigma\begin{pmatrix} A + BN & BM \\ L & K \end{pmatrix} \subset \mathbb{C}_s$$

if and only if $(I-PF)^{-1}P$, $(I-PF)^{-1}PF$, $(I-FP)^{-1}F$, *and* $(I-FP)^{-1}FP$ *are s-stable.*

Proof. A proof of this can be found in [13, p. 103]. ■

The latter result will be very important to us. It means that once we have a s-stabilizable and s-detectable candidate compensator (2.2) for one of the (almost) noninteracting-control problems we want to solve, we can check whether it makes the closed-loop system internally s-stable simply by checking whether the four transfer matrices appearing in Lemma 4.1 are s-stable. The typical compensator construction that will be used in this paper is the following. Suppose that (A, B) is s-stabilizable, and let X and U be s-stable, strictly proper real rational matrices satisfying

$$(Is-A)X(s)-BU(s)=I. \tag{4.3}$$

Since $sX(s)=I+BU(s)+AX(s)$, which is bicausal, the rational matrix $sX(s)$ has a proper inverse, say $L(s)$. Consequently also $X(s)$ is invertible with $X(s)^{-1}=sL(s)$ (not necessarily proper). Now define

$$F(s):=U(s)X(s)^{-1}. \tag{4.4}$$

We claim that F is *proper*. Indeed, this is easy, since $F(s)=sU(s)L(s)$ with $sU(s)$ proper and $L(s)$ proper. We contend that $F(s)$ is the transfer matrix of a stabilizing compensator:

LEMMA 4.2. *Assume that* (A, B) *is s-stabilizable. Let* X *and* U *be strictly proper s-stable real rational matrices such that* $(Is-A)X(s)-BU(s)=I$. *Let* $F(s):=U(s)X(s)^{-1}$. *Then for every realization* $N+M(Is-K)^{-1}L$ *of* $F(s)$ *such that* (K, L) *is s-stabilizable and* (M, K) *is s-detectable we have*

$$\sigma\begin{pmatrix} A+BN & BM \\ L & K \end{pmatrix} \subset \mathbb{C}_s.$$

Proof. By straightforward calculation it can be seen that

$$[I-P(s)F(s)]^{-1}P(s)=X(s)B,$$

$$[I-P(s)F(s)]^{-1}P(s)F(s)=X(s)(Is-A)-I,$$

$$[I-F(s)P(s)]^{-1}F(s)=U(s)(Is-A)$$

and finally that

$$[I-F(s)P(s)]^{-1}F(s)P(s)=U(s)B.$$

Since all these are s-stable, the conclusion follows from Lemma 4.1. ∎

We note that the result of Lemma 4.2 is strongly related to [13, Section 5.2], in the sense that (4.3) can be interpreted as a Bézout equation that yields a stabilizing compensator. In contrast with [13], however, our left-coprime factorization of the plant (4.1) is taken with respect to the ring of stable but not necessarily proper rational matrices.

5. NONINTERACTING CONTROL: MAIN RESULTS

In this section we shall formulate and prove necessary and sufficient conditions for solvability of Problem IV in terms of the subspaces that we considered in Section 3. Subsequently, as corollaries we shall state conditions for solvability of Problems I, II, and III.

Before starting off, we shall introduce one more important concept, the concept of *radical* [17]. Given a finite collection $\{L_i \mid i \in \underline{r}\}$ of subspaces of a linear space X, its radical is defined as the subspace

$$L_0 := \sum_{i=1}^{r}\left(L_i \cap \sum_{\substack{j \neq i\\ j=1}}^{r} L_j\right). \tag{5.1}$$

The collection $\{L_i \mid i \in \underline{r}\}$ is said to be *independent* if the radical L_0 is equal to $\{0\}$. For an extensive discussion on the various properties of the radical and its application to the "extended decoupling problem" we refer to [17]. In the sequel we shall make use of the following lemma:

LEMMA 5.1. *Let $\{L_i \mid i \in \underline{r}\}$ be a collection of subspaces of X. Let $\overline{L}_i \subset X_i$ be subspaces such that*

$$L_0 \oplus \overline{L}_i = L_0 + L_i, \qquad i \in \underline{r}. \tag{5.2}$$

Then the collection $\{L_0, \overline{L}_i \mid i \in \underline{r}\}$ is independent.

Proof. For a proof of this lemma we refer to Appendix A. ∎

Now consider the system (2.1) to be controlled. Denote $\operatorname{im} G_i$ by G_i. The radical of the collection $\{G_i \mid i \in \underline{k}\}$ will be denoted by G_0. Furthermore, define

$$K := \bigcap_{j=1}^{k} \ker D_j, \qquad K_i := \bigcap_{\substack{j \neq i \\ j=1}}^{k} \ker D_j, \quad i \in \underline{k}. \tag{5.3}$$

The following theorem is the main result of this section:

THEOREM 5.2. *Problem* IV *is solvable if and only if* (A, B) *is s-stabilizable,*

$$G_i \subset V_{f,s}(K_i, K) \qquad \textit{for all} \quad i \in \underline{k}, \tag{5.4}$$

and

$$G_0 \subset V_s^*(K). \tag{5.5}$$

In order to establish a proof of this result we shall proceed as follows. Define

$$M_j(s) := D_j(Is - A)^{-1}B,$$

$$N_i(s) := (Is - A)^{-1}G_i,$$

and

$$W_{ij}(s) := D_j(Is - A)^{-1}G_i,$$

the open-loop transfer matrices from u to z_j, v_i to x, and v_i to z_j respectively. Given a dynamic compensator (2.2), denote its transfer matrix by F. Let P be defined by (4.1). A straightforward calculation shows that the closed-loop transfer matrix from v_i to z_j [as defined by (2.4)] is equal to

$$T_{ij} = W_{ij} + M_j(I - FP)^{-1}FN_i, \tag{5.6}$$

and thus the solvability of Problem IV is equivalent to the existence of a proper real rational F such that $W_{ij} + M_j(I - FP)^{-1}FN_i = 0$ for all $i, j \in \underline{k}$ with $i \neq j$, such that $W_{ii} + M_i(I - FP)^{-1}FN_i$ is f-stable for all $i \in \underline{k}$ and such that the closed-loop system is internally s-stable. This brings us in a position to establish a proof of Theorem 5.2.

Proof of Theorem 5.2. $\Leftarrow$: Assume that (A, B) is s-stabilizable and that (5.4) and (5.5) hold. By Lemma 3.3, for all i there exist s-stable strictly proper real rational matrices X_i, U_i such that $(Is - A)X_i(s) - BU_i(s) = I$,

$$D_j X_i G_i = 0 \qquad \text{for all} \quad j \neq i, \tag{5.7}$$

and

$$D_i X_i G_i \text{ is } f\text{-stable}. \tag{5.8}$$

Moreover, there exist s-stable strictly proper real rational matrices X_0 and U_0 such that $(Is - A)X_0(s) - BU(s) = I$ and

$$D_j X_0 G_0 = 0 \qquad \text{for all } j. \tag{5.9}$$

Here, G_0 is any linear map such that $\mathcal{G}_0 = \operatorname{im} G_0$. Now, we shall construct s-stable strictly proper real rational matrices X and U such that $(Is - A)X(s) - BU(s) = I$,

$$D_j X G_i = 0 \qquad \text{for all } i, j \text{ with } i \neq j, \tag{5.10}$$

and

$$D_i X G_i \text{ is } f\text{-stable for all } i. \tag{5.11}$$

After doing this we shall argue that any s-stabilizable and s-detectable realization of the proper rational matrix $U(s)X(s)^{-1}$ gives us a compensator which establishes the requirements of Problem IV.

Our construction will be based on a suitable direct-sum decomposition of the state space X. For $i \in \underline{k}$ let $\overline{\mathcal{G}}_i \subset \mathcal{G}_i$ be subspaces such that $\overline{\mathcal{G}}_i \oplus \mathcal{G}_0 = \mathcal{G}_i + \mathcal{G}_0$. According to Lemma 5.1 the collection of subspaces $\{\mathcal{G}_0, \overline{\mathcal{G}}_i \mid \in \underline{k}\}$ is independent. Let $\mathcal{G}_{k+1}$ be any complement of $\mathcal{G}_0 + \Sigma_i \overline{\mathcal{G}}_i$ in X. This yields a direct-sum decomposition

$$X = \mathcal{G}_0 \oplus \overline{\mathcal{G}}_1 \oplus \overline{\mathcal{G}}_2 \oplus \cdots \oplus \overline{\mathcal{G}}_k \oplus \mathcal{G}_{k+1}. \tag{5.12}$$

Let $P_0 : X \to X$ be the projector onto $\mathcal{G}_0$ along $\Sigma_{i=1}^k \overline{\mathcal{G}}_i + \mathcal{G}_{k+1}$, let P_i be the projector onto $\overline{G}_i$ along the other members of the decomposition (5.12), and let P_{k+1} be the projector onto G_{k+1} along $G_0 + \Sigma_{i=1}^{k}\overline{G}_i$. Note that $\Sigma_i P_i = I$ and that $G_i \subset \ker P_j \cap \ker P_{k+1}$ for all $i, j \in \underline{k}$ with $i \neq j$.

Now let X_{k+1}, U_{k+1} be any pair of s-stable strictly proper real rational matrices satisfying $(Is - A)X_{k+1}(s) - BU_{k+1}(s) = I$, and define

$$X := \sum_{i=0}^{k+1} X_i P_i, \qquad U := \sum_{i=0}^{k+1} U_i P_i. \tag{5.13}$$

Then we have

$$(Is - A)X(s) - BU(s) = \sum_{i=0}^{k+1} \left[(Is - A)X_i(s) - BU_i(s) \right] P_i = I,$$

which shows that (4.3) holds for the X, U defined by (5.13). Moreover, for all $i, j \in \underline{k}$ we have

$$D_j X G_i = D_j \left(\sum_{l=0}^{k+1} X_l P_l \right) G_i = D_j X_0 P_0 G_i + D_j X_i P_i G_i. \tag{5.14}$$

Now note that, since $\operatorname{im} P_0 G_i \subset G_0$, the first term on the right of (5.14) vanishes. Thus, since $\operatorname{im} P_i G_i \subset G_i$, we conclude from (5.7) and (5.8) that $D_j X G_i = 0$ for $i \neq j$ and $D_i X G_i$ is $\mathcal{f}$-stable. Define $F := UX^{-1}$. As already noted in the proof of Lemma 4.2, we then have

$$U(s) = [I - F(s)P(s)]^{-1} F(s)(Is - A)^{-1}.$$

Also, from $(Is - A)X(s) - BU(s) = I$, we have

$$X(s) = (Is - A)^{-1} + (Is - A)^{-1} BU(s).$$

Combining these two expressions we immediately obtain

$$D_j X G_i = W_{ij} + M_j (I - FP)^{-1} F N_i = T_{ij}.$$

We conclude that *any* realization $N + M(Is - K)^{-1}L$ of $X(s)U(s)^{-1}$ defines a compensator that achieves noninteraction with input-output $\mathcal{f}$-stability. Finally, since X and U are s-stable, it follows from Lemma 4.2 that in addition any such realization with (K, L) s-stabilizable and (M, K) s-detectable yields an internally s-stable closed loop system.

$\Rightarrow$: Assume Problem IV is solvable, i.e., there exists a compensator (2.2) such that $D_{j,e}(Is - A_e)^{-1}G_{i,e} = 0$ for all i, j with $i \neq j$, $D_{i,e}(Is - A_e)^{-1}G_{i,e}$ is f-stable for all i and $\sigma(A_e) \subset \mathbb{C}_s$. We will first show that $G_i \subset V_{f,s}(K_i, K)$. For every $v \in V_i$ define

$$\begin{pmatrix} \xi(s) \\ \nu(s) \end{pmatrix} := (Is - A_e)^{-1}\begin{pmatrix} G_i \\ 0 \end{pmatrix} v. \tag{5.15}$$

Then it is immediate that $G_i v = (Is - A)\xi(s) - B[M\nu(s) + N\xi(s)]$, which is a regular s-stable (ξ, ω) representation of $G_i v$. Since also $D_j\xi = 0$ for all $j \neq i$ and $D_i\xi$ is f-stable (so consequently $D_j\xi$ is f-stable for all j), we conclude that $G_i v \in V_{f,s}(K_i, K)$.

Next, we shall prove that the radical $G_0 \subset V_s{}^*(K)$. By (5.1), it suffices to show that $G_i \cap \Sigma_{j \neq i} G_j \subset V_s{}^*(K)$ for all i. Let $G_i v = \Sigma_{j \neq i} G_j w_j$. We contend that $D_{j,e}(Is - A_e)^{-1}G_{i,e}v = 0$ for all j. Now, for $j \neq i$ this is immediate. On the other hand,

$$D_{i,e}(Is - A_e)^{-1}G_{i,e}v = \sum_{j \neq i} D_{i,e}(Is - A_e)^{-1}G_{j,e}w_j = 0,$$

which proves our claim. Now define ξ and ν by (5.15). Then $G_i v = (Is - A)\xi(s) - B[M\nu(s) + N\xi(s)]$, which is a regular s-stable (ξ, ω) representation of $G_i v$ such that $D_j\xi = 0$ for all j. It follows that $G_i v \in V_s{}^*(K)$. ■

We stress that, since $V_{f,s}(K_i, K) = V_f{}^*(K_i) + V_s{}^*(K)$ (see Theorem 3.2), the conditions established above can indeed be checked constructively. An actual check would involve the calculation of k f-stabilizability subspaces $V_f{}^*(K_i)$ and of the s-stabilizability subspace $V_s{}^*(K)$. A conceptual algorithm for this is described in [17, p. 114].

REMARK 5.3. A few words on the dynamic order of the compensator as constructed in the proof of Theorem 5.2 are in order here. Using the fact that X and U are related by $(Is - A)X(s) - BU(s)$, it may be shown that the McMillan degrees of the compensator UX^{-1} and the rational matrix X respectively are related by $\deg(UX^{-1}) \leqslant \deg(X) - n$. (See Appendix B, Lemma B.1.) Thus, in order to obtain an upper bound to the McMillan degree of the compensator, it is of interest to obtain such a bound for X. Now, it can be shown that the X_i in terms of which X is defined [see (5.13)] can in fact be constructed in such a way that we have $\deg(X_i P_i) \leqslant$

$\dim V_{f,s}(K_i, K)$ ($i \in \underline{k}$) and such that $\deg(X_0 P_0 + X_{k+1} P_{k+1}) \leqslant n$. Consequently, if Problem IV is solvable, then a required compensator (2.2) exists with dynamic order satisfying

$$\dim W \leqslant \sum_{i=1}^{k} \dim V_{f,s}(K_i, K). \tag{5.16}$$

Note that this upper bound increases as the number of input-output channels to be decoupled increases.

As already noted in Section 2, the main theorem (Theorem 5.2) immediately provides necessary and sufficient conditions for solvability of the simpler Problems I, II, and III:

COROLLARY 5.4. (i) *Problem* III *is solvable if and only if* (A, B) *is g-stabilizable,* $G_i \subset V_g^*(K_i)$ *for all* $i \in \underline{k}$, *and* $G_0 \subset V_g^*(K)$.

(ii) *Problem* II *is solvable if and only if* $G_i \subset V_g^*(K_i) + V^*(K)$ *for all* $i \in \underline{k}$ *and* $G_0 \subset V^*(K)$.

(iii) [17] *Problem* I *is solvable if and only if* $G_i \subset V^*(K_i)$ *for all* $i \in \underline{k}$ *and* $G_0 \subset V^*(K)$. ■

We conclude this section by noting that in certain situations it is desirable instead of a proper compensator to design a *strictly* proper compensator Σ_c that achieves noninteraction. Indeed, using the methods developed here, it can for example be shown that there exists a compensator (2.2) with $N = 0$ such that $T_{ij} = 0$ for $i \neq j$ if and only if $G_i + AG_i \subset V^*(K_i)$ for $i \in \underline{k}$ and $G_0 + AG_0 \subset V^*(K)$.

6. ALMOST NONINTERACTING CONTROL: PROBLEM FORMULATION

If instead of requiring the off-diagonal blocks in the closed-loop transfer matrix to be exactly equal to zero we only require these blocks to be arbitrarily small in some appropriate norm, we arrive at problems in the context of approximate or "almost" noninteracting control. In the present section we shall formulate the "almost" analogues of the synthesis problems we studied in the foregoing. In the following the magnitudes of the closed-loop transfer matrices involved will always be measured in H^∞-norm. Let $\mathbb{C}^-$ denote $\{s \in \mathbb{C} \mid \operatorname{Re} s < 0\}$. Given a $\mathbb{C}^-$-stable proper real rational $p \times m$

matrix W, its H^∞-norm is defined as

$$\|W\|_\infty := \sup_{\operatorname{Re} s \geqslant 0} \|W(s)\|.$$

Here, for $s \in \mathbb{C}$, $\|W(s)\|$ denotes the operator norm of the complex matrix $W(s)$ considered as a linear map from $\mathbb{C}^m$ to $\mathbb{C}^p$ or, equivalently, the largest singular value of the complex matrix $W(s)$. For more details we refer to [13] or [3].

Consider the system (2.1). If we require the off-diagonal blocks in the closed-loop transfer matrix to have arbitrarily small H^∞-norm, we arrive at the following problem: for all $\varepsilon > 0$ determine a compensator (2.2) such that $\|T_{ij}\|_\infty \leqslant \varepsilon$ for all $i \neq j$. In this form it is required for all $\varepsilon > 0$ to find a suitable compensator state space W_ε together with suitable linear maps K_ε, L_ε, M_ε, and N_ε. Now, in practice one would like to exclude the possibility that as ε becomes smaller and smaller (i.e. as the decoupling accuracy increases), the dynamic order $\dim W_\varepsilon$ of the compensator increases unboundedly. Therefore, we shall require not only that the off-diagonal blocks in the closed loop transfer matrix can be made arbitrarily small, but in addition that this can be done without having to increase the dynamic order of the required compensators unboudedly. In this way, denoting the dynamic order $\dim W$ of the compensator (2.2) by $\dim \Sigma_c$, we arrive at the following formulation:

Problem V (Almost noninteracting control). Problem V is said to be solvable if there exists an integer N and if for all $\varepsilon > 0$ there exists a compensator Σ_c with $\dim \Sigma_c \leqslant N$ such that $\|T_{ij}\|_\infty \leqslant \varepsilon$ for all $i, j \in \underline{k}$ with $i \neq j$.

If, apart from approximate noninteraction up to any desired degree of accuracy, we require input-output stability of the closed-loop system with respect to a given stability set $\mathbb{C}_g$, we arrive at:

Problem VI (Almost noninteracting control with input-output-stability). Given a symmetric subset $\mathbb{C}_g$ of $\mathbb{C}$, Problem VI is said to be solvable if there exists an integer N and if for all $\varepsilon > 0$ there exists a compensator Σ_c with $\dim \Sigma_c \leqslant N$ such that $\|T_{ij}\|_\infty \leqslant \varepsilon$ for all $i, j \in \underline{k}$ with $i \neq j$ and such that T_{ij} is g-stable for all $i, j \in \underline{k}$.

Note that by requiring $\|T_{ij}\|_\infty \leqslant \varepsilon$ for $i \neq j$ it is of course already implicitly assumed that T_{ij} is $\mathbb{C}^-$-stable for $i \neq j$. Thus, in the particular case that in the above we take $\mathbb{C}_g$ equal to $\mathbb{C}^-$, the requirement "T_{ij} is g-stable for all $i, j \in \underline{k}$" can be replaced by "T_{ii} is g-stable for all $i \in \underline{k}$" without changing the problem. Also note that a necessary condition for Problem VI to be solvable is that $\mathbb{C}_g \cap \mathbb{C}^- \neq \varnothing$, a condition that will of course be satisfied for any reasonable choice of $\mathbb{C}_g$.

If instead of input-output stability we require *internal* stability of the closed-loop system, we obtain the following:

PROBLEM VII (Almost noninteracting control with internal stability). Given a symmetric subset $\mathbb{C}_g$ of $\mathbb{C}$, Problem VII is said to be solvable if there exists an integer N and if for all $\varepsilon > 0$ there exists a compensator Σ_c with $\dim \Sigma_c \leqslant N$ such that $\|T_{ij}\|_\infty \leqslant \varepsilon$ for all $i, j \in \underline{k}$ with $i \neq j$ and such that $\sigma(A_e) \subseteq \mathbb{C}_g$.

Finally, by combining the requirements of internal stability and input-output stability into one synthesis problem we can formulate:

PROBLEM VIII (Almost noninteracting control with input-output and internal stability). Given two symmetric subsets $\mathbb{C}_f \subset \mathbb{C}_s$ of $\mathbb{C}$, Problem VIII will be said to be solvable if there exists an integer N and if for all $\varepsilon > 0$ there exists a compensator Σ_c with $\dim \Sigma_c \leqslant N$ such that $\|T_{ij}\|_\infty \leqslant \varepsilon$ for all $i, j \in \underline{k}$ with $i \neq j$, T_{ij} is f-stable for all $i, j \in \underline{k}$, and $\sigma(A_e) \subset \mathbb{C}_s$.

In the sequel again we shall concentrate on the last of these four problems, Problem VIII, as the first three can be obtained from this one as special cases.

7. ALMOST INVARIANT SUBSPACES

Given a system (A, B) and a subspace $K = \ker H$ of the state space X, we shall denote by $V_b^*(K)$ the supremal L_1-almost controlled invariant subspace of K, and by $R_b^*(K)$ the supremal L_1-almost controllability subspace of K. For the exact definitions and extensive treatments of these subspaces, see [14], [15], and [12].

In the following, a subset $\mathbb{C}_g$ of $\mathbb{C}$ will be said to *contain minus infinity* if it has the property that there exists $c \in \mathbb{R}$ such that $(-\infty, c] \subset \mathbb{C}_g$. In the context of almost invariant subspaces the latter is a natural assumption to make on stability sets (see also [9] and [12]). The family of all symmetric subsets of $\mathbb{C}$ that contain minus infinity will be denoted by S_∞. We recall that for a given proper rational matrix or vector X its *McMillan degree* is denoted by $\deg(X)$. In [12] the following characterizations in terms of regular (ξ, ω)

representations were established:

PROPOSITION 7.1.

$$V_b^*(K) = \{ x_0 \in X \mid \textit{ for all } \varepsilon > 0 \textit{ there is a regular } (\xi, \omega) \textit{ representation of } x_0 \textit{ with } \|H\xi\|_\infty \leqslant \varepsilon \}.$$

$$R_b^*(K) = \{ x_0 \in X \mid \textit{there is } r \in \mathbb{N} \textit{ and for all } \varepsilon > 0 \textit{ and for every } \mathbb{C}_g \in \mathrm{S}_\infty \textit{ there is a regular g-stable } (\xi, \omega) \textit{ representation of } x_0 \textit{ with } \deg(\xi) \leqslant r \textit{ such that } \|H\xi\|_\infty \leqslant \varepsilon \}.$$

Proof. See [12, Corollary 3.33] and [12, Corollary 3.37]. ■

We stress that in the above the upper bound r to the McMillan degrees of the ξ's is allowed to depend on x_0 but is independent of ε. In the time domain, loosely speaking, the above states that $R_b^*(K)$ is equal to the subspace of X with the property that starting in it one may travel along trajectories such that their distance to K is arbitrarily small and their characteristic values are located arbitrarily in the complex plane.

As in our previous considerations on the exact noninteracting-control problem, in the sequel an important role will be played by almost controlled invariant subspaces that are defined in terms of pairs of stability sets and pairs of subspaces of the state space. Let $\mathbb{C}_f \subset \mathbb{C}_s$ be symmetric subsets of $\mathbb{C}$, and let $\ker H_2 = K_2 \subset K_1 = \ker H_1$ be subspaces of X. The following definition is the "almost" analogue of the subspace $V_{f,s}(K_1, K_2)$ as defined in Section 3:

DEFINITION 7.2.

$$W_{f,s}(K_1, K_2) := \{ x_0 \in X \mid \text{there is } r \in \mathbb{N} \text{ and for all } \varepsilon > 0 \text{ there is a regular } s\text{-stable } (\xi, \omega) \text{ representation of } x_0 \text{ with } \deg(\xi) \leqslant r \text{ such that } \|H_1\xi\|_\infty \leqslant \varepsilon \text{ and } H_2\xi \text{ is } f\text{-stable} \}.$$

It is clear that the above defines a linear subspace of X, which by Proposition 7.1 is contained in $V_b^*(K_1)$. It also follows immediately from the definitions that $V_{f,s}(K_1, K_2) \subset W_{f,s}(K_1, K_2)$. In addition, if $\mathbb{C}_f \in S_\infty$ then we have $R_b^*(K_1) \subset W_{f,s}(K_1, K_2)$. This follows by taking $\mathbb{C}_g = \mathbb{C}_f$ in the characterization of $R_b^*(K_1)$ given in Proposition 7.1 and by noting that $\mathbb{C}_f \subset \mathbb{C}_s$. Thus we find that if $\mathbb{C}_f \in S_\infty$ then

$$V_s^*(K_2) + V_f^*(K_1) + R_b^*(K_1) \subset W_{f,s}(K_1, K_2). \tag{7.1}$$

It will be proven in Section 8 that under some fairly mild additional assumptions on the set $\mathbb{C}_f$ the inclusion (7.1) is in fact an equality. Since the three subspaces on the left in (7.1) can be calculated using simple algorithms (see [17, p. 114] and [15]), this means that for those $\mathbb{C}_f$'s we will actually be able to *calculate* explicitly the subspace $W_{f,s}(K_1, K_2)$. Consequently, we will also be able to check every subspace inclusion involving $W_{f,s}(K_1, K_2)$ constructively. Keeping the latter fact in mind as a motivation, we now state the following analogue of Lemma 3.3:

LEMMA 7.3. *Let* E *be a linear subspace of* X *and let* E *be a linear map such that* $\operatorname{im} E = \mathsf{E}$. *Then* (A, B) *is s-stabilizable and* $\mathsf{E} \subset W_{f,s}(K_1, K_2)$ *if and only if there exists* $r \in \mathbb{N}$ *and for all* $\varepsilon > 0$ *there exist strictly proper s-stable real rational matrices* X *and* U *with* $\deg(X) \leqslant r$ *such that* $(Is - A)X(s) - BU(s) = I$, $\|H_1XE\|_\infty \leqslant \varepsilon$, *and* H_2XE *is f-stable.*

Proof. This follows immediately from Definition 7.2 and can be proven completely analogously to Lemma 3.3. ■

In addition to Definition 7.2, for given symmetric subsets $\mathbb{C}_f \subset \mathbb{C}_s$ of $\mathbb{C}$ and a given single subspace $K = \ker H$ of X we define

$$W_{f,s}(K) := W_{f,s}(K, K).$$

It follows from Lemma 7.3 that (A, B) is s-stabilizable and $\operatorname{im} E \subset W_{f,s}(K)$ if and only if there exists $r \in \mathbb{N}$ and for all $\varepsilon > 0$ there exist strictly proper s-stable real rational matrices X and U with $\deg(X) \leqslant r$ such that $(Is - A)X(s) - BU(s) = I$, $\|HXE\|_\infty \leqslant \varepsilon$, and HXE is f-stable.

We shall now return to the almost noninteracting-control problem. Consider the system (2.1), and let K and K_i be defined by (5.3). Again, let G_0 denote the radical of the family of subspaces $\{G_i \mid i \in \underline{k}\}$. The following result provides necessary and sufficient conditions for solvability of Problem VIII in terms of the subspaces introduced in this section:

THEOREM 7.4. *Problem* VIII *is solvable if and only if* (A, B) *is s-stabilizable*,

$$G_i \subset W_{f,s}(K_i, K) \qquad \text{for all} \quad i \in \underline{k}, \tag{7.2}$$

and

$$G_0 \subset W_{f,s}(K). \tag{7.3}$$

Proof. This can be proven in an entirely analogous way to its "exact" version, Theorem 5.2. As in the proof of Theorem 5.2, the idea is to apply Lemma 7.3 to each of the $k+1$ subspace inclusions (7.2) and (7.3) and to "glue together" (this time for each ε) the X_i's and U_i's into one pair of rational matrices X and U in order to obtain a compensator UX^{-1} (depending of course on ε). A detailed proof is given in Appendix B. ■

Before we continue, we want to stress that the above theorem is of course of little use unless we find a way to express the subspaces $W_{f,s}(K_i, K)$ and $W_{f,s}(K)$ in terms of subspaces that can in principle be *calculated*. As already announced in this section, under some mild assumptions on the stability set $\mathbb{C}_f$ it turns out that these subspaces can indeed be characterized as the sums of stabilizability subspaces and L_1-almost controllability subspaces. Since such characterization is independent of the noninteracting control context, the main importance of Theorem 7.4 is that it reduces our main problem to a problem of obtaining a satisfactory characterization of the single subspace $W_{f,s}(K_1, K_2)$ defined in Definition 7.2. The latter will be the subject of Section 8.

8. ALMOST NONINTERACTING CONTROL: MAIN RESULTS

In the present section we shall establish conditions under which the subspace inclusion (7.1) (which was shown to hold under the assumption that $\mathbb{C}_f \in \mathbb{S}_\infty$) can be replaced by equality. Again consider a system (A, B), and let $\ker H_2 = K_2 \subset K_1 = \ker H_1$ be subspaces of the state space X. Let $\mathbb{C}_f \subset \mathbb{C}_s$ be symmetric subsets of $\mathbb{C}$. In the following, let $\overline{\mathbb{C}}_f$ denote the topological closure of $\mathbb{C}_f$ in $\mathbb{C}$. Furthermore, let $\sigma^*(A, B, H_1)$ denote the set of *invariant zeros* associated with the system (A, B, H_1)—i.e., the fixed spectrum $\sigma(A + BF|V^*(K_1)/R^*(K_1))$ (see [17, p. 112] or [1]). We shall show that the inclusion (7.1) is in fact an equality if the following assumption holds: $\mathbb{C}_f$ is symmetric and contains minus infinity, and $\overline{\mathbb{C}}_f \backslash \mathbb{C}_f$ contains no invariant zeros of (A, B, H_1). Note in particular that if $\mathbb{C}_f \in \mathrm{S}_\infty$ is closed, then the above will of course trivially hold.

Before going into the details, we shall first give an example of a situation in which the inclusion (7.1) is strict:

EXAMPLE 8.1.

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$K_1 = \ker H_1 \quad \text{with } H_1 = (0 \quad 1), \qquad K_2 = \ker I_{2\times 2}.$$

Let $\mathbb{C}_f = \{ s \in \mathbb{C} \mid \operatorname{Re} s < -1 \}$, and let $\mathbb{C}_s$ be any symmetric subset of $\mathbb{C}$ containing $\mathbb{C}_f$. Note that $\mathbb{C}_f \in \mathrm{S}_\infty$. Using the algorithms in [17, p. 114] and [15], we calculate that

$$V_f{}^*(K_1) = \{0\}, \qquad V_s{}^*(K_2) = \{0\}, \qquad R_b^*(K_1) = \operatorname{im} B.$$

We claim however that

$$x_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

lies in $W_{f,s}(K_1, K_2)$. To see this, let $0 < \varepsilon < 1$ and define a state feedback by

$$F_\varepsilon := \left(-\tfrac{1}{4}\varepsilon^2, -\varepsilon \right).$$

Define $\xi_\varepsilon(s) := (Is - A - BF_\varepsilon)^{-1} x_0$ and $\omega_\varepsilon(s) := F_\varepsilon \xi_\varepsilon(s)$. This yields a regular (ξ, ω) representation of x_0 that obviously has the property that $\deg(\xi_\varepsilon) \leqslant 2$. Moreover, $\sigma(A + BF_\varepsilon) = \{ -1 - \frac{1}{2}\varepsilon, -1 - \frac{1}{2}\varepsilon \} \subset \mathbb{C}_f$, and therefore ξ_ε is s-stable and $H_2 \xi_\varepsilon = \xi_\varepsilon$ is f-stable. Finally, we calculate that

$$\|H_1 \xi_\varepsilon\|_\infty = \frac{\frac{1}{4}\varepsilon^2}{1 + \varepsilon + \frac{1}{4}\varepsilon^2} \leqslant \tfrac{1}{4}\varepsilon^2 < \varepsilon$$

and conclude that $x_0 \in W_{f,s}(K_1, K_2)$. It can be shown that $s = 1$ is an invariant zero of (A, B, H_1). This invariant zero is contained in $\overline{\mathbb{C}}_f \backslash \mathbb{C}_f = \{ s \in \mathbb{C} \mid \operatorname{Re} s = 1 \}$.

In order to proceed, let $W_f(K) := W_{f,f}(K, K)$ denote the subspace obtained by taking $\mathbb{C}_s = \mathbb{C}_f$ and $K_1 = K_2 = K$ in Definition 7.2. Consequently, if $K = \ker H$, then we have

$$W_f(K) = \{x_0 \in X \mid \text{there is } r \in \mathbb{N} \text{ and for all } \varepsilon > 0 \text{ there exists a regular } f\text{-stable } (\xi, \omega) \text{ representation of } x_0 \text{ with } \deg(\xi) \leqslant r \text{ such that } \|H\xi\|_\infty \leqslant \varepsilon\}. \tag{8.1}$$

The following lemma reduces the problem of finding an in principle computable expression for $W_{f,s}(K_1, K_2)$ to the problem of finding such an expression for $W_f(K_1)$. Since this subspace depends on only one stability set and one subspace, the latter is expected to be easier.

LEMMA 8.2. $W_{f,s}(K_1, K_2) = V_s^*(K_2) + W_f(K_1)$.

Proof. $\subset$: Let $x_0 \in W_{f,s}(K_1, K_2)$, and let $r \in \mathbb{N}$ be the integer associated with x_0. Let $\varepsilon > 0$. There are strictly proper s-stable rational vectors ξ and ω with $\deg(\xi) \leqslant r$ such that $x_0 = (Is - A)\xi(s) - B\omega(s)$, $\|H_1\xi\|_\infty \leqslant \varepsilon$, and $H_2\xi$ is f-stable. Since ξ and ω are strictly proper, they can be decomposed uniquely as $\xi = \xi_1 + \xi_2$ and $\omega = \omega_1 + \omega_2$, where ξ_1, ω_1 are strictly proper and f-stable and ξ_2, ω_2 are strictly proper and $\mathbb{C}_s \backslash \mathbb{C}_f$-stable. Consequently we have

$$x_0 - (Is - A)\xi_1(s) + B\omega_1(s) = (Is - A)\xi_2(s) - B\omega_2(s),$$

where the left-hand side is proper and $\mathbb{C}_f$-stable and the right-hand side is proper and $\mathbb{C}_s \backslash \mathbb{C}_f$-stable. This however implies that both sides must in fact be equal to the same constant vector, say x_{02}. This yields

$$x_{02} = (Is - A)\xi_2(s) - B\omega_2(s),$$

$$x_0 - x_{02} = (Is - A)\xi_1(s) - B\omega_1(s).$$

We also have $H_2\xi_2 = H_2\xi - H_2\xi_1$. The right-hand side of this equation is $\mathbb{C}_f$-stable; the left-hand side is $\mathbb{C}_s \backslash \mathbb{C}_f$-stable. Consequently, $H_2\xi_2 = 0$. Since also ξ_2 is s-stable, it follows from (3.2) that $x_{02} \in V_s^*(K_2)$. As for $x_0 - x_{02}$, note that since $K_2 \subset K_1$ we have $H_1\xi_2 = 0$. Hence we have

$$\|H_1\xi_1\|_\infty = \|H_1\xi - H_1\xi_2\|_\infty = \|H_1\xi\|_\infty \leqslant \varepsilon.$$

Since also ξ_1 is f-stable and since $\deg(\xi_1) \leqslant \deg(\xi) \leqslant r$ it follows that $x_0 - x_{02} \in W_f(K_1)$.

$\supset$: The converse inclusion follows immediately from (3.2) and the definition of $W_f(K_1)$. ■

Motivated by the above lemma, we shall concentrate on studying the subspace given by (8.1). Let $K = \ker H$ be a subspace of X, and let $\mathbb{C}_f$ be a symmetric subset of $\mathbb{C}$. By (3.2) it is immediate that $V_f^*(K) \subset W_f(K)$. In addition, from Proposition 7.1, it follows that if $\mathbb{C}_f$ is symmetric and contains minus infinity, then $R_b^*(K) \subset W_f(K)$. Thus, if $\mathbb{C}_f \in S_\infty$ then we have

$$V_f^*(K) + R_b^*(K) \subset W_f(K). \tag{8.2}$$

Note that by combining Lemma 8.2 and (8.2) we reobtain (7.1). Also note that Example 8.1 provides an example of a situation in which the inclusion in (8.2) is strict (take $K = K_1$).

In the sequel, if $\mathbb{C}_f$ is a given stability set, then $V_{\bar{f}}^*(K)$ will denote the supremal stabilizability subspace in K with respect to $\overline{\mathbb{C}}_f$ (the closure of $\mathbb{C}_f$). We have the following:

Lemma 8.3. *Assume that $\mathbb{C}_f$ is a symmetric subset of $\mathbb{C}$. Then*

$$W_f(K) \subset V_{\bar{f}}^*(K) + R_b^*(K). \tag{8.3}$$

Proof. For a proof of this lemma we refer to Appendix C. ■

By combining the two inclusions obtained above, we immediately see that if $V_f^*(K) = V_{\bar{f}}^*(K)$, then the inclusion (8.2) is in fact an equality. It is well known that the structure of stabilizability subspaces is closely connected to the notion of invariant zero. In fact, we obtain:

Corollary 8.4. *Let $\mathbb{C}_f \in S_\infty$. Assume that $\sigma^*(A, B, H) \cap (\overline{\mathbb{C}}_f \setminus \mathbb{C}_f) = \varnothing$. Then we have $W_f(K) = V_f^*(K) + R_b^*(K)$.*

Proof. Let $R^*(K)$ be the supremal controllability subspace in K. Let F be such that $(A + BF)V^*(K) \subset V^*(K)$. Denote $\sigma^*(A, B, H)$ by σ^*. Denote by E_s the generalized eigenspace of the mapping $(A + BF)|V^*(K)$ associated with the eigenvalue s. By [17, p. 114], for a given symmetric subset $\mathbb{C}_g$ of $\mathbb{C}$

we have

$$V_g^*(K) = R^*(K) \oplus \left(\bigoplus_{s \in \sigma^* \cap \mathbb{C}_g} E_s \right).$$

Since $\sigma^* \cap \mathbb{C}_f = \sigma^* \cap \bar{\mathbb{C}}_f$, this implies that $V_f^*(K) = V_{\bar{f}}^*(K)$. ∎

Briefly summarizing the above, we see that in general we do not have equality in (8.2). A counterexample was provided by Example 8.1. However, if we make an assumption on the position of the invariant zeros, we do obtain equality in (8.2). In particular, if we assume that $\mathbb{C}_f$ is closed, then this assumption will always be satisfied. Another possibility to make sure that the assumption holds for a given system is to choose $\mathbb{C}_f$ "sufficiently far too the left" in the open left half plane. By applying Corollary 8.4 to the case that $\mathbb{C}_f = \mathbb{C}^-$, we obtain that in this case equality holds in (8.2) if (A, B, H) has no invariant zeros on the imaginary axis.

Collecting the above results, the following is now immediate:

Corollary 8.5. *Consider the system* (A, B) *and let* $\ker H_2 = K_2 \subset K_1 = \ker H_1$ *be subspaces of the state space* X. *Let* $\mathbb{C}_f \subset \mathbb{C}_s$ *be symmetric subsets of* $\mathbb{C}$. *Assume that the following conditions are satisfied*:

$$\mathbb{C}_f \in S_\infty \quad \textit{and} \quad \sigma^*(A, B, H_1) \cap \left(\bar{\mathbb{C}}_f \backslash \mathbb{C}_f \right) = \varnothing.$$

Then we have

$$W_{f,s}(K_1, K_2) = V_s^*(K_2) + V_f^*(K_1) + R_b^*(K_2).$$

Let us now return to the almost noninteracting-control problem. Consider the system (2.1), and again let K and K_i be defined by (5.3). Let G_0 be the radical of the G_i's. At this point we have all material needed to obtain the following results on the solvability of Problems V to VIII:

Corollary 8.6. *Assume that the following conditions hold*: $\mathbb{C}_f \in S_\infty$ *and is closed. Then Problem* VIII *is solvable if and only if* (A, B) *is s-stabilizable*,

$$G_i \subset V_f^*(K_i) + V_s^*(K) + R_b^*(K_i) \qquad \textit{for all} \quad i \in \underline{k},$$

and

$$G_0 \subset V_s^*(K) + R_b^*(K).$$

COROLLARY 8.7. *Assume that $\mathbb{C}_g \in S_\infty$ and is closed. Then we have:*

(i) *Problem* VII *is solvable if and only if* (A, B) *is g-stabilizable,* $G_i \subset V_g^*(K_i) + R_b^*(K_i)$ *for all* $i \in \underline{k}$, *and* $G_0 \subset V_g^*(K) + R_b^*(K)$.

(ii) *Problem* VI *is solvable if and only if* $G_i \subset V_g^*(K_i) + V^*(K) + R_b^*(K_i)$ *for all* $i \in \underline{k}$ *and* $G_0 \subset V^*(K) + R_b^*(K)$.

(iii) *Problem* V *is solvable if and only if* $G_i \subset V^*(K_i) + R_b^*(K_i)$ *for all* $i \in \underline{k}$ *and* $G_0 \subset V^*(K) + R_b^*(K)$.

REMARK 8.8. For the sake of simplicity, in the statement of the above corollaries we have chosen a closedness condition on the stability sets. Alternatively, however, it is possible to formulate a more general condition involving the invariant zeros. In fact, if we define

$$\hat{D}_i := \left(D_1^T, D_2^T, \dots, D_{i-1}^T, D_{i+1}^T, \dots, D_k^T\right)^T, \qquad i \in \underline{k},$$

and

$$\hat{D} := \left(D_1^T, D_2^T, \dots, D_k^T\right),$$

then it can be seen that for all i we have $\sigma^*(A, B, \hat{D}_i) \subset \sigma^*(A, B, \hat{D})$. Using this fact, it is easy to show that the statement of Corollary 8.7 remains valid if we replace the condition by

$$\sigma^*(A, B, \hat{D}) \cap \left(\overline{\mathbb{C}}_f \setminus \mathbb{C}_f\right) = \varnothing.$$

REMARK 8.9. Also in the "almost" case it is possible to establish an upper bound to the required order of compensation. Under the assumptions of Corollary 8.7 it is possible to show that for every decoupling accuracy ε a compensator (2.2) can be found with dynamic order satisfying

$$\dim W \leqslant \sum_{i=1}^{k} \dim\left[V_{f,s}(K_i, K) + \langle A | \operatorname{im} B\rangle\right].$$

Here $\langle A | \operatorname{im} B\rangle$ denotes the reachable subspace of (A, B). As was required in the definition of Problem VIII, this upper bound does not depend on the decoupling accuracy ε.

9. CONCLUDING REMARKS

In this paper we have been able to find solvability conditions for two rather general problems in the context of noninteracting control by dynamic state feedback. The first of these was a problem of exact block-diagonal decoupling with internal stability and input-output stability, the second its "almost" analogue in which only approximate decoupling was required. As special cases we obtained conditions for solvability of the corresponding problems where only input-output stability, only internal stability, or no stability was required.

There are several points that we did not consider in this paper. One interesting problem would be to find conditions for solvability of the problems treated here with an additional requirement of output controllability preservation (see also [5]). In our context, preservation of output trajectories would mean that we would restrict the class of admissible compensators to those compensators that have the property that the diagonal blocks do not lose rank. More concretely, for each diagonal block the normal rank after compensation should at least be equal to the rank of the corresponding block before compensation.

Another interesting problem would be to generalize the above theory to the case of dynamic *measurement* feedback. At this point, however, even for the exact noninteracting control problem without any stability requirements such extension seems to be a very hard problem.

APPENDIX A

Proof of Lemma 5.1. We shall first show that the collection $\{\bar{L}_i \mid i \in \underline{r}\}$ is independent. Assume the contrary. Then there is an index i such that

$$\bar{L}_i \cap \sum_{j \neq i} \bar{L}_i \neq \{0\}. \tag{A.1}$$

Now, on the one hand the subspace (A.1) is contained in $\bar{L}_i$. On the other hand it is contained in the radical of the collection $\{L_0 + L_i \mid i \in \underline{r}\}$, which, by [18, Lemma 10.1], is equal to L_0. Thus, (A.1) is contained in $\bar{L}_i \cap L_0 = \{0\}$, which is a contradiction. To complete the proof it suffices to show that $L_0 \cap (\bar{L}_i \oplus \bar{L}_2 \oplus \cdots \oplus \bar{L}_k) = \{0\}$. Assume that $x = \Sigma_j x_j \in L_0$ with $x_j \in \bar{L}_j$. Then we have $\Sigma_{j \neq i} x_j \in L_0 \oplus \bar{L}_i = L_0 + L_i$. Also, $\Sigma_{j \neq i} x_j \in \Sigma_{j \neq i}(L_0 + L_j)$ and consequently $\Sigma_{j \neq i} x_j$ is an element of the radical of $\{L_0 + L_j \mid j \in \underline{r}\}$. Since the latter equals L_0, we find that $\Sigma_{j \neq i} x_j \in L_0$. However, since

$\Sigma_j x_j = x \in L_0$ it follows that $x_i \in L_0$. We conclude that $x_i \in \overline{L}_i \cap L_0 = \{0\}$, whence $x_i = 0$. The latter argument holds for every $i \in \underline{k}$, and therefore $x = 0$. This completes the proof of the lemma. ■

APPENDIX B

In this appendix we will give a proof of Theorem 7.4. First we shall prove the following useful result (see also Remark 5.3). Recall that for a given proper rational matrix X its McMillan degree is denoted by $\deg(X)$.

LEMMA B.1. *Consider the system (A, B). Let X and U be strictly proper real rational matrices such that $(Is - A)X(s) - BU(s) = I$. Then we have $\deg(UX^{-1}) \leqslant \deg(X) - n$.*

Proof. Let $F(s) := U(s)X(s)^{-1}$. Since $Is - A - BF(s) = X(s)^{-1}$ and since B is injective, we have $\deg(F) \leqslant \deg(X^{-1})$. We claim that $\deg(X^{-1}) = \deg(X) - n$. Indeed, this follows immediately upon comparing the Smith-McMillan forms of X and X^{-1} respectively and using the fact that $sX(s)$ is bicausal. ■

Proof of Theorem 7.4. $\Leftarrow$: Let X be decomposed according to (5.12), and for $i = 0, 1, \dots, k+1$ let P_i be the projectors associated with this direct-sum decomposition. Let G_0 be a linear map such that $\mathcal{G}_0 = \operatorname{im} G_0$. Since $\operatorname{im} P_i \subset \mathcal{G}_i$ $(i = 0, 1, \dots, k+1)$, there are linear maps T_i such that $P_i G_i = G_i T_i$. Assume now that the hypotheses of Theorem 7.4 hold. Then by Lemma 7.3, for all $i \in \underline{k}$ there is $r_i \in \mathbb{N}$ such that for all $\varepsilon > 0$ there are s-stable strictly proper real rational matrices X_i and U_i with $\deg(X_i) \leqslant r_i$, $(Is - A)X_i(s) - BU_i(s) = I$,

$$\|D_j X_i G_i\|_\infty \leqslant \frac{\varepsilon}{2(\|T_i\| + 1)} \qquad \text{for all} \quad j \in \underline{k}, \quad j \neq i, \tag{B.1}$$

and

$$D_j X_i G_i \ s\text{-stable for all } j \in \underline{k}. \tag{B.2}$$

Also, there is $r_0 \in \mathbb{N}$ such that for all $\varepsilon > 0$ there are s-stable strictly proper real rational matrices X_0 and U_0 with $\deg(X_0) \leqslant r_0$, $(Is - A)X_0(s) - BU_0(s) = I$,

$$\|D_j X_0 G_0\|_\infty \leqslant \frac{\varepsilon}{2(\|T_0\| + 1)} \qquad \text{for all} \quad j \in \underline{k}, \tag{B.3}$$

and

$$D_j X_0 G_0 \text{ } \mathcal{f}\text{-stable for all } j \in \underline{k}. \tag{B.4}$$

Now, let $\varepsilon > 0$, and let X_i, U_i be such that the above conditions are satisfied. Choose arbitrary s-stable strictly proper real rational matrices X_{k+1} and U_{k+1} such that $(Is - A)X_{k+1}(s) - BU_{k+1}(s) = I$. Let $r_{k+1} := \deg(X_{k+1})$. As in the proof of Theorem 5.2, define

$$X := \sum_{i=0}^{k+1} X_i P_i, \qquad U := \sum_{i=0}^{k+1} U_i P_i.$$

Then X and U are s-stable, $(Is - A)X(s) - BU(s) = I$, and $\deg(X) \leqslant \Sigma_i r_i =: N$ (independent of ε). Moreover, for all $i, j \in \underline{k}$ with $i \neq j$ it follows from (B.1) and (B.3) that

$$\begin{aligned}\|D_j X G_i\|_\infty &= \|D_j X_0 P_0 G_i + D_j X_i P_i G_i\|_\infty \\ &\leqslant \|D_j X_0 G_0\|_\infty \|T_0\| + \|D_j X_i G_i\|_\infty \|T_i\| < \varepsilon.\end{aligned}$$

Also, for all $i, j \in \underline{k}$ we have

$$D_j X G_i = D_j X_0 G_0 T_0 + D_j X_i G_i T_i,$$

which by (B.2) and (B.4) is $\mathcal{f}$-stable. Define now a compensator transfer matrix by $F := UX^{-1}$. By the previous lemma we have $\deg(F) \leqslant N$. Moreover, as in the proof of Theorem 5.2, for all $i, j \in \underline{k}$, $D_j X G_i$ is equal to T_{ij}, the closed-loop transfer matrix from v_i to z_j. Consequently, any minimal realization Σ_c of $F(s)$ yields $\|T_{ij}\|_\infty \leqslant \varepsilon$ $(i \neq j)$ and makes T_{ij} $\mathcal{f}$-stable for all i, j. Since X and U are s-stable, by Lemma 4.4 Σ_c also yields an internally s-stable closed loop system. Finally, $\dim \Sigma_c = \deg(F) \leqslant N$ (independent of ε).

$\Rightarrow$: The converse implication of the theorem is also proven completely analogously to the corresponding proof of Theorem 5.2. The proof is left to the reader. ∎

APPENDIX C

In this appendix a proof will be established of Lemma 8.3. Until so far, all relevant subspaces have been characterized in terms of *regular* (ξ, ω) representations. For most of the subspaces appearing in Sections 7 and 8, however, characterizations in terms of not necessarily strictly proper (ξ, ω) representations can also be given. In the time domain such representations correspond to distributional state trajectories and controls (see also [15] and [12]). In the following, consider the system (A, B) and let K be a subspace of X. The following characterization will be useful:

LEMMA C.1. *Let* $\mathbb{C}_f$ *be a symmetric subset of* $\mathbb{C}$. *Then*

$$V_f^*(K)+R_b^*(K)=\{x_0\in X \mid x_0 \textit{ has an } f\textit{-stable } (\xi,\omega) \textit{ representation with } H\xi=0\}.$$

For a proof of this we refer to [9, Proposition 2.10] (see also [8]). Now, the idea of the proof of Lemma 8.3 that we will give here is as follows. Given $x_0 \in W_f(K)$, by definition there is an integer r and there are sequences (ξ_n) and (ω_n) of f-stable strictly proper rational vectors with $\deg(\xi_n) \leqslant r$ such that $\|H\xi_n\|_\infty \to 0$ $(n \to \infty)$ and $x_0 = (Is - A)\xi_n(s) - B\omega_n(s)$. The idea is to analyze the limiting behavior of the sequences (ξ_n) and (ω_n) and to produce not necessarily strictly proper rational vectors ξ and ω that in a certain sense are limits of the ξ_n's and ω_n's for $n \to \infty$. For these vectors we will have $x_0 = (Is - A)\xi(s) - B\omega(s)$. Moreover we will have $H\xi = 0$ and, since the ξ_n's are f-stable, it will turn out that ξ is $\overline{\mathbb{C}}_f$-stable. It then follows from the previous lemma that $x_0 \in V_f^*(K) + R_b^*(K)$. In the sequel we shall elaborate this idea.

Given a real rational function $f = p/q$ with p and q coprime polynomials, define the degree of f as $\partial(f) := \max(\partial(p), \partial(q))$. Here $\partial(p)$ and $\partial(q)$ are the usual degrees of p and q as polynomials. If f is proper, then $\partial(f)$ coincides with the McMillan degree $\deg(f)$. By $\sigma(f)$ we shall denote the set of poles of f (i.e. the zeros of the denominator q). The following result states that if (f_n) is a sequence of rational functions of which the degrees are uniformly bounded from above and if f_n converges to a rational function f pointwise for infinitely many $s \in \mathbb{C}$, then the poles of f lie in the closure of the union of the poles of all f_n's:

LEMMA C.2. *Let* $(f_n)_{n \in \mathbb{N}}$ *be a sequence of real rational functions. Assume there exists* $r \in \mathbb{N}$ *such that* $\partial(f_n) \leqslant r$ *for all* $n \in \mathbb{N}$. *Assume that* f *is* *a real rational function such that* $f_n(s) \to f(s)$ $(n \to \infty)$ *for infinitely many* $s \in \mathbb{C}$. *Then*

$$\sigma(f) \subset \overline{\bigcup_{n=1}^{\infty} \sigma(f_n)}\,.$$

Proof. Using the existence of an upper bound r to the degrees $\partial(f_n)$ and the pointwise convergence for infinitely many $s \in \mathbb{C}$, it was shown in [6, p. 169–171] that

(i) for every $n \in \mathbb{N}$ there exist coprime polynomials $p_n(s) = \sum_{i=0}^{r} a_i(n) s^i$ and $q_n(s) = \sum_{i=0}^{r} b_i(n) s^i$ such that $f_n(s) = p_n(s) q_n(s)^{-1}$;

(ii) there exist polynomials $p(s) = \sum_{i=0}^{i} a_i s^i$ and $q(s) = \sum_{i=0}^{i} b_i s^i$ such that $f(s) = p(s) q(s)^{-1}$;

(iii) the coefficients satisfy $a_j(n) \to a_j$ and $b_j(n) \to b_j$ $(n \to \infty)$ for all $j \in \underline{r}$.

Since the coefficients of the polynomials q_n converge to those of q, it may be shown that $q_n \to q$ *uniformly* on compact subsets of $\mathbb{C}$. Now, let $s_0 \in \sigma(f)$. Then $q(s_0) = 0$. Let $\mu > 0$ be such that s_0 is the only zero of q in the disc $|s - s_0| < \mu$. Take an arbitrary $\varepsilon \in (0, \mu)$. We shall prove that the disc $|s - s_0| < \varepsilon$ contains an element of $\sigma(f_N)$ for some sufficiently large N.

Define

$$\alpha := \min\{\,|q(s)|\,\big|\,|s - s_0| = \varepsilon\}.$$

Then $\alpha > 0$. Since $q_n \to q$ uniformly on the circle

$$C_\varepsilon := \{s \in \mathbb{C}\,\big|\,|s - s_0| = \varepsilon\},$$

there exists a $N \in \mathbb{N}$ such that $n \geqslant N$ implies $|q_n(s) - q(s)| < \alpha$ for all s on the circle C_ε. Now define a polynomial $g := q_N - q$. Then $|g(s)| = |q_N(s) - q(s)| < \alpha \leqslant |q(s)|$ on C_ε. Hence, by Rouché's theorem (see [2, p. 116]), $g + q = q_N$ has a zero *inside* the disc $|s - s_0| < \varepsilon$. Since q_N and p_N are coprime, this zero is a pole of f_N and hence lies in $\sigma(f_N)$.

We are now in a position to give a proof of Lemma 8.3. In the following, let $R^*(K)$ be the supremal controllability subspace contained in K (see [17, p. 109]).

Proof of Lemma 8.3. Let $x_0 \in W_f(K)$. There is $r \in \mathbb{N}$ and there are sequences (ξ_n) and (ω_n) of strictly proper real rational vectors with ξ_n

f-stable, $\deg(\xi_n) \leqslant r$, $\|H\xi_n\|_\infty \to 0$ $(n \to \infty)$, and $x_0 = (Is - A)\xi_n(s) - B\omega_n(s)$. Let $F: X \to U$ be such that $(A+BF)R^*(K) \subset R^*(K)$ and $\sigma(A+BF)|R^*(K) \subset \mathbb{C}_f$. Define $\bar{\omega}_n(s) := \omega_n(s) - F\xi_n(s)$. Denote $A+BF$ by A_F. Then we have

$$x_0 = (Is - A_F)\xi_n(s) - B\bar{\omega}_n(s) \qquad \forall n.$$

Make a direct-sum decomposition $X = X_1 \oplus X_2$ with $X_1 := R^*(K)$ and X_2 any complement of X_1 in X. With respect to a basis compatible with this decomposition we have

$$A_F = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad \text{and} \quad H = \begin{pmatrix} 0 & H_2 \end{pmatrix}.$$

Furthermore, $\sigma(A_{11}) \subset \mathbb{C}_f$. Next, make a direct-sum decomposition $U = U_1 \oplus U_2$ with $U_1 := \ker B_2$ and U_2 any complement of U_1 in U. With respect to this decomposition, let $B_1 = (B_{11}\ \ B_{12})$ and $B_2 = (0\ \ B_{22})$. Obviously B_{22} is injective. By applying [17, Exercise 4.4] and [17, Exercise 5.8] it can be seen that the system (A_{22}, B_{22}, H_2) is left-invertible in the sense that the transfer matrix $T(s) := H_2(Is - A_{22})^{-1}B_{22}$ has a left inverse $T^+(s)$. Partition

$$x_0 = \begin{pmatrix} x_{01} \\ x_{02} \end{pmatrix}, \qquad \bar{\omega}_n = \begin{pmatrix} \bar{\omega}_{1n} \\ \bar{\omega}_{2n} \end{pmatrix}, \quad \text{and} \quad \xi_n = \begin{pmatrix} \xi_{1n} \\ \xi_{2n} \end{pmatrix}.$$

Then we have

$$x_{02} = (Is - A_{22})\xi_{2n}(s) - B_{22}\bar{\omega}_{2n}(s),$$

which yields

$$\bar{\omega}_{2n}(s) = T^+(s)\left[H_2\xi_{2n}(s) + H_2(Is - A_{22})^{-1}x_{02}\right].$$

Now, obviously $\|H_2\xi_{2n}\|_\infty = \|H\xi_n\|_\infty \to 0$ as $n \to \infty$, and consequently for all $s \in \mathbb{C}^+ = \{s \in \mathbb{C} \mid \operatorname{Re} s \geqslant 0\}$ we have

$$\|H_2\xi_{2n}(s)\| \leqslant \|H_2\xi_{2n}\|_\infty \to 0 \qquad (n \to \infty).$$

A fortiori this implies that $\bar{\omega}_{2n}(s) \to T^+(s)H_2(Is - A_{22})^{-1}x_{02} =: \bar{\omega}_2(s)$ for infinitely many $s \in \mathbb{C}$. Since

$$\xi_{2n}(s) = (Is - A_{22})^{-1}(B_{22}\bar{\omega}_{2n}(s) + x_{02}) \qquad \forall n,$$

this implies that $\xi_{2n}(s) \to (Is - A_{22})^{-1}[B_{22}\bar{\omega}_2(s) + x_{02}] =: \xi_2(s)$ for infinitely many $s \in \mathbb{C}$. Note that $\bar{\omega}_2$ and ξ_2 are rational but not necessarily strictly proper. Also note that $H_2\xi_2 = 0$ and that $x_{02} = (Is - A_{22})\xi_2(s) - B_{22}\bar{\omega}_2(s)$. Since ξ_{2n} is f-stable for all n and since the degrees of all its components are bounded from above, it follows from Lemma C.2 that ξ_2 has all its poles in the closure $\overline{\mathbb{C}}_f$. The same holds for $\bar{\omega}_2$. Define now

$$\bar{\omega} := \begin{pmatrix} 0 \\ \bar{\omega}_2 \end{pmatrix} \quad \text{and} \quad \xi_1(s) := (Is - A_{11})^{-1}[B_{12}\bar{\omega}_2(s) + A_{12}\xi_2(s)].$$

Since $\sigma(A_{11}) \subset \mathbb{C}_f$, ξ_1 is f-stable. Moreover,

$$\begin{pmatrix} 0 \\ x_{02} \end{pmatrix} = \begin{pmatrix} Is - A_{11} & -A_{12} \\ 0 & Is - A_{22} \end{pmatrix}\begin{pmatrix} \xi_1(s) \\ \xi_2(s) \end{pmatrix} - \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}\begin{pmatrix} 0 \\ \bar{\omega}_2(s) \end{pmatrix}.$$

The latter expression is a (ξ, ω) representation of $\begin{pmatrix} 0 \\ x_{02} \end{pmatrix}$ with $\xi := (\xi_1^T, \xi_2^T)^T$ $\overline{\mathbb{C}}_f$-stable and $H\xi = 0$. Consequently, it follows from Lemma C.1 that the vector $\begin{pmatrix} 0 \\ x_{02} \end{pmatrix}$ lies in $V_f^*(K) + R_b^*(K)$. Since

$$\begin{pmatrix} x_{01} \\ 0 \end{pmatrix} \in R^*(K) \subset R_b^*(K)$$

(see e.g. [15]), this yields $x_0 \in V_f^*(K) + R_b^*(K)$. This completes the proof of the lemma. ■

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